Tensor Products of Division Algebras and Fields

Louis Rowen
Department of Mathematics
Bar-Ilan University
Ramat Gan, 52900, Israel

David J. Saltman
Center for Communications Research
Princeton, NJ 08540

Abstract This paper began as an investigation of the question of whether $D_1 \otimes_F D_2$ is a domain where the D_i are division algebras and F is an algebraically closed field contained in their centers. We present an example where the answer is "no", and also study the Picard group and Brauer group properties of $F_1 \otimes_F F_2$ where the F_i are fields. Finally, as part of our example, we have results about division algebras and Brauer groups over curves. Specifically, we give a splitting criterion for certain Brauer group elements on the product of two curves over F.

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Introduction

This paper was first motivated by the following question 0.1, posed some time ago by M. Schacher, cf. [Sc], and resurrected more recently by L. Small:

Question 0.1. *Suppose that D_1 and D_2 are division algebras (finite over their centers), and both centers contain an algebraically closed field F. Does $D_1 \otimes_F D_2$ have zero divisors?*

We have an example where 0.1 is false, which appears in Section 4.

Let F_i be the center of D_i and write this as D_i/F_i. Then it is well known (and we reprove below) that $F_1 \otimes_F F_2$ is a domain and we can set $K = q(F_1 \otimes_F F_2)$ to be its field of fractions. Now Question 0.1 is equivalent to asking whether $(D_1 \otimes_{F_1} K) \otimes_K (D_2 \otimes_{F_2} K)$ is a division algebra. We will frequently switch between these two points of view. It is enough to consider Question 0.1 when the D_i/F_i have prime power degree, with respect to the same prime number. We will always assume this, and frequently restrict our focus to the case where the D_i have (the same) prime degree. This assumption simplifies our arguments, and still is challenging and interesting.

As work on this problem proceeded, it became clear that this should be viewed as a piece of the following more general subject. To understand Question 0.1 one needs to understand the center of $D_1 \otimes_F D_2$ which leads to:

Question 0.2. *Suppose F is algebraically closed and F_i/F are field extensions. What are the properties of $F_1 \otimes_F F_2$?*

Obviously Question 0.2 is ridiculously vague, but in this paper we will ask and partially answer questions about the Picard groups and Brauer groups of $F_1 \otimes_F F_2$. This seems most relevant to Question 0.1. Moreover, Question 0.1 can be viewed as being subsumed by Question 0.2 if we include in Question 0.2 the properties of Azumaya algebras with center $F_1 \otimes_F F_2$.

In Question 0.1 and every version of Question 0.2 we consider here, we may reduce to the case that the D_i/F_i are finitely generated **as division algebras**. That is, the F_i/F are finitely generated as fields. In other language, we write $F_i = F(V_i)$ for a projective variety V_i.

In Sections 2 through 4 below, we will assume that the ground field F has characteristic 0. This is to allow us to quote resolution of singularities and write $F_i = F(V_i)$ where V_i is a non-singular projective variety. In Section 3 we quote and use the theorem of resolution of divisors.

Let us outline the paper to follow. In the rest of this introduction we define some notation and observe one well known general Brauer group fact. In Section 1 we make general observations about Question 0.1, including the affirmative answer when D_1 is commutative. Also in Section 1 is the perhaps surprising connection between Question 0.1 and the ramification of the D_i. More precisely, if D_1 is totally ramified at a discrete valuation domain R with $q(R) = F_1$, then $D_1 \otimes_F D_2$ is a domain. In particular, when the D_i have prime degree, and D_1 is ramified, then $D_1 \otimes_F D_2$ is a domain. This is partial justification for the idea that "usually" $D_1 \otimes_F D_2$ is a domain.

In Section 2 we cover some results about ramification that we need in Section 3. The main idea is to show that we can eliminate all ramification of a Brauer group element with a finite field extension. In our case we need this extension to be a tensor product of extensions of each of the $F_i = F(V_i)$. Though we do not need it here, it is natural to ask that this extension be of degree bounded in terms of the order of the Brauer group element and the dimensions of the V_i. This stronger result is due to A. Pirutka [P] and we repeat a slight modification of her argument here so we can further observe that the extension can be chosen a product a fields as we need.

Section 3 contains the main body of our results about $F_1 \otimes_F F_2$. Our feeling is that a full understanding of Question 0.1 in general requires a fuller understanding of these rings. Finally, Section 4 has our example where $D_1 \otimes_F D_2$ is not a domain, accompanied by the theory of Brauer groups over curves that we need for the example. This material on Brauer groups of curves has obvious independent interest.

The following fact is well known but a reference is hard to find. If V is a scheme, then $\mathrm{Br}(W)'$ for any $W \to V$ is defined to be the subgroup of the Brauer group comprised of all elements of order prime to the characteristic of all field points of V.

Proposition 0.3. *Let V be an irreducible non-singular scheme. Then $\mathrm{Br}(V)' = \cap_{P \subset V} \mathrm{Br}(\mathcal{O}_{V,P})'$ where P ranges over all irreducible codimension 1 subschemes and $\mathcal{O}_{V,P}$ is the stalk of V at P, so $\mathcal{O}_{V,P}$ is a discrete valuation domain.*

Proof. When $V = \mathrm{Spec}(R)$ for a regular local ring R this is [Ho]. In [M, p. 147] it is observed that $U \to \mathrm{Br}(U)'$ is a Zariski sheaf and the result follows. ■

Section 1. First Observations

In this section we make some initial observations about question 0.1. It is well known that if both $D_i = F_i$ are commutative then $F_1 \otimes_F F_2$ is an integral domain. The next result is a generalization.

Lemma 1.1. *Suppose $D_1 = F_1$ is commutative. Then $F_1 \otimes_F D_2$ has no zero divisors.*

Proof. Write $F_1 = F(V_1)$. Suppose $0 \neq \alpha_i = \sum_j a_{i,j} \otimes b_{i,j} \in F_1 \otimes_F D_2$ are such that $\alpha_1\alpha_2 = 0$. We can assume that for each i the set of $b_{i,j}$ are linearly independent over F. Since V_1 is irreducible, there is an F point on V_1 such that all the $a_{i,j}$ are defined at this point and for each i one of the $a_{i,j}$ is nonzero. That is, there is a local ring $R \subset F_1$ and $\phi : R \to F$ such that all $a_{i,j} \in R$, and for each i some $\phi(a_{i,j}) \neq 0$. Then ϕ induces $\Phi : R \otimes_F D_2 \to F \otimes_F D_2 = D_2$ such that $\Phi(\alpha_i)$ is defined and $\Phi(\alpha_1)\Phi(\alpha_2) = 0$. Since the $b_{i,j}$ are linearly independent we have $\Phi(\alpha_i) \neq 0$ for both i, and this is a contradiction. ■

Note that this argument uses very little about D_2 except that it is a domain. In particular, we need not assume that it is finite over its center. Secondly, if F_1 is arbitrary (i.e. not necessarily finitely generated over its center) we can replace it

by the subfield generated by the $a_{i,j}$ and the same result holds. All of this initially suggested to us that $D_1 \otimes_F D_2$ is always a domain, but we have a counterexample. However, our intuition still is the (not precise) feeling that $D_1 \otimes_F D_2$ is **usually** a domain.

Though 1.1 is not hard, when combined with valuations, it yields a result that says that $D_1 \otimes_F D_2$ very often is a domain. Let R be a discrete valuation domain with fraction field $q(R) = F_1$. Then R defines on (most of) the Brauer group $\mathrm{Br}(F_1)$ a ramification map $\mathrm{ram}_R : \mathrm{Br}(F_1)' \to H^1(\bar{R}, \mathbb{Q}/\mathbb{Z})$. We say that D_1 is ramified at R if $\mathrm{ram}_R([D_1]) \neq 0$. We say that D_1 is totally ramified if the order of $\mathrm{ram}_R([D_1])$ is equal to the degree of D_1. Note that if D_1 has prime degree, then it is ramified at R if and only if it is totally ramified at R.

To be unramified at all possible R is a very strong condition. We define the unramified Brauer group to be the intersection of the kernels of all these ramification maps, with respect to all these R. The unramified Brauer group is much much smaller than the full Brauer group. Thus the following result suggests that $D_1 \otimes_F D_2$ is "generically" a domain.

Theorem 1.2. *a) Suppose D_1 is totally ramified at some discrete valuation domain R with $q(R) = F_1$. Then $D_1 \otimes_F D_1$ is a domain.*

b) Suppose D_1/F_1 has prime degree p not equal to the characteristic of F. If D_1 is ramified with respect to some R as in part (a), then $D_1 \otimes_F D_2$ is a domain.

Proof. Part b) is a consequence of part a) by our remark above. To prove part a), let $\hat{R}$ be the completion and $\hat{F} = q(\hat{R})$. Denote by $\bar{F}_1 = R/M = \hat{R}/\hat{M}$ the residue field of R and $\hat{R}$. Our assumption on D_1 implies that $D_1 \otimes_F \hat{F}$ has degree equal to exponent and thus is a division algebra. It follows that R extends to a noncommutative discrete valuation ring $S \subset D_1$ which defines a valuation on D_1. More precisely, S contains a unique maximal ideal $S\pi$ such that $S\pi = \pi S$ is a two sided ideal, $D_1^* = \cup_{n \in Z} S^* \pi^n = \cup_n \pi^n S^*$, and $S, \pi S$ lies over R, M. Since D_1 is totally ramified, it follows that $L = S/\pi S$ is a (commutative) field.

Suppose $0 \neq \alpha_i \in D_1 \otimes_F D_2$ are such that $\alpha_1 \alpha_2 = 0$. Write $\alpha_i = \sum a_{i,j} \otimes b_{i,j}$ as above, where, again, for each i the $b_{i,j}$ are linearly independent over F. Then we can write all $a_{1,j} = \pi^{m_j} u_{1,j}$ and all $a_{2,j} = u_{2,j} \pi^{n_j}$ where all $u_{i,j} \in S^*$. By changing α_1 into $\pi^n \alpha_1$ for some n we can assume all $m_j \geq 0$ and some $m_j = 0$. Similarly, working on the other side, we can assume all $n_j \geq 0$ and some $n_j = 0$. In our other language, if $\phi : S \to L$ is the canonical morphism, we have all $\phi(a_{i,j})$ are defined and for each i there is j such that $\phi(a_{i,j}) \neq 0$. Again $\phi : S \to L$ induces $\Phi : S \otimes_F D_2 \to L \otimes_F D_2$ and $\Phi(\alpha_i) = \sum_j \phi(a_{i,j}) \otimes_F b_{i,j} \in L \otimes_F D_2$. Since the $b_{i,j}$ are linearly independent over F the $1 \otimes b_{i,j} \in L \otimes_F D_2$ must be linearly independent over L. In particular, $\Phi(\alpha_i) \neq 0$ for both i. Since $\Phi(\alpha_1)\Phi(\alpha_2) = 0$ we have a contradiction to Lemma 1.1. ∎

We remark that again 1.2 uses nothing about D_2 except that it is a domain. We also remark that the totally ramified condition can be eased a bit. Suppose D_1 is ramified at such an R and $D_1 \otimes_F \hat{F}$ is a division algebra. Let S exists as above but $S/\pi S = \bar{D}_1$ is a division algebra with center $L \not\supset \bar{F}_1$. Note that $\bar{D}_1/L$

has degree smaller than D_1 and the proof of 1.2 shows that if $D_1 \otimes_F D_2$ has a zero divisor then so does $\bar{D}_1 \otimes_F D_2$.

Section 2. Ramification

In this section we investigate some questions about ramification of Brauer group elements that we need in the rest of the paper. To be precise, we need a result that any Brauer group element $\alpha \in \mathrm{Br}(F(V_1) \otimes_F F(V_2))$ restricts to an everywhere unramified element after an extension of the form $F(V_1' \times_F V_2')$ where each $F(V_i')/F(V_i)$ is a finite field extension. This can be done, but our original proof of this had the uncomfortable property that the degree of this extension is unbounded as we vary α among all elements of the same order. This unboundedness did not constrain our arguments here, but was unsatisfactory as it seemed that there should be a bound on the degrees of the $F(V_i')/F(V_i)$ that only depends on the dimension of the V_i and the order of α.

In fact, ignoring the specific requirements of this paper, the more natural question is the following. Suppose $\alpha \in \mathrm{Br}(F(V))$. Is there a field extension $F(V')/F(V)$ splitting all the ramification of α, with degree bounded by a function of the order of α and the dimension of V? It is believed that the **index** of α should have a similar bound. The result about splitting ramification would then be evidence for this index conjecture. The second author asked the above splitting ramification question at the workshop "Deformation Theory, Patching, Quadratic Forms, and the Brauer Group" in January 2011 at the American Institute of Mathematics. In April 2011 an affirmative answer was provided by Alena Pirutka [P]. Pirutka's result also generalizes to higher degree cohomology. In this section we provide a slightly modified proof of her result because we need to observe further that we can choose our $F(V')/F(V)$ to have the form $F(V_1' \times_F V_2')/F(V_1 \times V_2)$, and there is no reason to give our earlier unbounded result. It should be noted that the bound in [P] (and below) is known not to be strict even in the dimension 2 case.

To accomplish these results we need to make an observation about what it takes to split all the ramification over a regular local ring. Let R be a regular local ring containing F, and take $\alpha \in \mathrm{Br}(q(R))$ of order q. Suppose the ramification locus of α has non-singular components with normal crossings at R.

Lemma 2.1. *Suppose α ramifies at $\pi \in M - M^2$ where M is the maximal ideal of R. Set $S = R/\pi$ and let $\bar{L}/q(S)$ be the ramification defined by α. Then all ramification of $\bar{L}/q(S)$ is at primes which are the images of primes in the ramification locus of α.*

Proof. Suppose not, and that in fact $\bar{L}$ ramifies at a prime $\bar{\delta}$ of S not on the list. Consider the inverse image of $\bar{\delta}$ which is a height two prime $Q \subset R$. Note that Q contains none of the primes, except π, where α ramifies. Let T be the localization of R at Q, so T is a two dimensional regular local ring. Set $\bar{T} = T/\pi$. Then π is the only prime of T where α ramifies. By e.g. [S, p. 129] this implies that $\bar{L}/q(\bar{T})$ is unramified at $\bar{T}$, a contradiction. ∎

Theorem 2.2. *If α and R are as above, and $\pi_1, \ldots, \pi_r$ are the primes where α ramifies, then $\alpha = [\prod_{j=r}^{1}(u_j \pi_1^{a_{1,j}} \ldots \pi_{j-1}^{a_{j-1,j}}, \pi_j)_q]\alpha'$, where the u_j are units and $\alpha' \in \mathrm{Br}(R)$.*

Proof. We induct on r. Let $\bar{R} = R/\pi_r$, and $\bar{L}/q(\bar{R})$ be the ramification of α at π_r. By the lemma $\bar{L}$ only ramifies on the images, $\bar{\pi}_i$, of the π_i for $i < r$. Since $\bar{R}$ is a UFD, this implies $\bar{L} = q(\bar{R})((\bar{u}_r \bar{\pi}_1^{a_{1,r}} \ldots \pi_{r-1}^{a_{r-1,r}})^{1/q})$ for a unit u_r and integers $a_{i,r}$. Thus, $\alpha/(u_r \pi_1^{a_{1,r}} \ldots \pi_{r-1}^{a_{r-1,r}}, \pi_r)_q$ does not ramify at π_r and only ramifies at the π_i for $i < r$. We are done by induction on r. ∎

We are going to kill ramification by the following trick.

Proposition 2.3. *Suppose R is a regular local ring and α and the π_j are as above. Let $L \supset q(R)$ be a field extension where for each i there are units v_i such that $v_i^{-1}\pi_i$ is an n power in L. Then α_L is unramified with respect to any discrete valuation lying over a localization of R.*

Proof. Write α as above and consider $\alpha'' = \alpha' \prod_{j=1}^{r}(u_j v_1^{a_{1,j}} \ldots v_{j-1}^{a_{j-1,j}}, v_j)_q$. Then α and α'' have the same image in $\mathrm{Br}(L)$ and $\alpha'' \in \mathrm{Br}(R)$. Thus $\alpha'' \in \mathrm{Br}(R_P)$ for any prime P and the result is clear. ∎

Of course the difficulty is in constructing such an L that works at all the stalks. As above, let V/F be smooth projective of dimension d and let $\alpha \in \mathrm{Br}(F(V))$. After blowing up (e.g. [K, p.138]) we may assume that the ramification locus of α consists of non-singular irreducible components with normal crossings. Our next result is really about such a set of divisors.

Theorem 2.4. *Let $\mathcal{D}$ be a set of non-singular irreducible divisors of V with normal crossings. Then there is a morphism $V' \to V$ formed by blowing up along a succession of non-singular subvarieties with the following property. Let $\mathcal{D}'$ be the set of divisors of V' formed as the union of strict transforms of all elements of $\mathcal{D}$ and all exceptional divisors (and the strict transforms of exceptional divisors). Then $\mathcal{D}'$ is the disjoint union of $\mathcal{D}'_i$ for $1 \le i \le d$ such that each $\mathcal{D}'_i$ consists of disjoint irreducible divisors.*

Proof. We will make repeated use of the following fact which uses that the components of $\mathcal{D}$ are all non-singular with normal crossings. Let E be a component of a non-trivial intersection of $D_1, \ldots, D_r$, all of which are elements of $\mathcal{D}$. If $V' \to V$ is the blowup at E and we identify D_i with its strict transform in V', then in V' the intersection of the D_i is empty.

Furthermore, any nonempty intersection of r elements of $\mathcal{D}$ has dimension $d - r$ and is the disjoint union of non-singular components of that dimension. In particular, $r \le d$. First we look at all the nonempty intersections of d elements of $\mathcal{D}$, which altogether comprise a finite set of points. We form $V_1 \to V$ by blowing up at all those points. Let $\mathcal{D}'_1 = \mathcal{D} \cup \mathcal{D}_1$, where $\mathcal{D}$ are the strict transforms in V_1 of the divisors $\mathcal{D}$ in V, and $\mathcal{D}_1$ are all the exceptional divisors which are obviously all disjoint.

In $\mathcal{D}$ (viewed as divisors in V_1) we define E_1 to be all nonempty intersections of subsets of order $d - 1$. Since any d elements of $\mathcal{D}$ have empty intersection, the

components of E_1 are all disjoint, non-singular curves. We let $V_2 \to V_1$ be the blowup at all these curves and set $\mathcal{D}_2' = \mathcal{D}_2 \cup \mathcal{D}_1 \cup \mathcal{D}$ where $\mathcal{D}_2$ are the exceptional divisors and the rest of the terms again are strict transforms. Proceeding in this way we are done because at the last step all the elements of $\mathcal{D}$ will be disjoint. ■

The above argument says that at the level of divisors we can separate the ramification locus of $\alpha \in \mathrm{Br}(F(V))$ so that if all the $\sum_{E \in \mathcal{D}_i} E$ were principal, we could take all these q roots and kill all ramification. Since these divisors need not be principal, we have to proceed as follows.

In the arguments to come, we will be given a finite set of irreducible closed sets $\mathcal{C}$ of a variety V'. Let $\mathcal{C}'$ be the union of $\mathcal{C}$ and the finite set of all components of all intersections of subsets of $\mathcal{C}$. Of course, $\mathcal{C}'$ is a finite set closed under the process of taking components of intersections of subsets. Let $\mathcal{M}$ be the set of minimal elements of $\mathcal{C}'$, being all elements which do not properly contain another element of $\mathcal{C}'$. Then all the elements of $\mathcal{M}$ are disjoint. Thus we can take the stalk of the structure sheaf $\mathcal{O}_V$ at $\mathcal{M}$ and by abuse of terminology we call this the stalk of $\mathcal{C}$.

Lemma 2.5. *Let R be the stalk of $\mathcal{C}$. Then R is a semilocal domain. If V' is non-singular, then R is regular and a UFD. The prime elements (up to units) of R correspond to all irreducible divisors of V which contain a component of an intersection of a subset of $\mathcal{C}$.*

Proof. That R is a UFD is well known and can be found, for example, in [S1, p. 1546]. This rest is all clear. ■

We return to a set of irreducible divisors $\mathcal{D}$ as in the conclusion of 2.4. That is, the elements of $\mathcal{D}$ are all non-singular and together they have normal crossings. Moreover, $\mathcal{D} = \cup_{i=1}^{d} \mathcal{D}_i$ where the irreducible divisors in each $\mathcal{D}_i$ are all disjoint. Let $\mathcal{D}_i$ consist of divisors $D_{i,j}$ where $1 \leq j \leq s(i)$. Let R_1 be the stalk of V at $\mathcal{D}$ which for the purposes of this argument we rename $\mathcal{E}_1$. Then all the $D_{i,j}$ induce primes $\pi_{i,j,1}$ on R_1 and we can choose $f_{i,1}$ such that $f_{i,1}R_1 = (\prod_j \pi_{i,j,1})R_1$. Looking globally, the principal divisor $(f_{i,1})$ equals $\sum_j D_{i,j} + \sum_k n_{i,k,1}E_{i,k,1}$, where no component of any of the intersections of any subset of the elements of $\mathcal{E}_1$ is contained in any $E_{i,k,1}$.

By induction, assume R_l, $\mathcal{E}_l$, $f_{i,l}$, $E_{i,k,l}$ and $n_{i,k,l}$ have been defined for all $l < m$ where:

a) $\mathcal{E}_l$ is the set of all $D_{i,j}$ and all $E_{i,k,l'}$ for all $l' < l$.

b) $(f_{i,l}) = \sum_j D_{i,j} + \sum_k n_{i,k,l}E_{i,k,l}$.

c) R_l is the stalk of $\mathcal{E}_l$.

d) No $E_{i,k,l}$ contains a component of an intersection of elements of $\mathcal{E}_l$.

Of course we define $\mathcal{E}_m$ to be the set of all $D_{i,j}$ and all $E_{i,k,l}$ for all i, k and $l < m$. Equally obviously, we set R_m to be the stalk at $\mathcal{E}_m$ and in R_m we let $D_{i,j}$ define $\pi_{i,j,m}$ on R_m. Let $f_{i,m}$ be such that $f_{i,m}R_m = (\prod_j \pi_{i,j,m})R_m$. Of course, we define the $E_{i,k,m}$ and $n_{i,k,m}$ via $(f_{i,m}) = \sum_j D_{i,j} + \sum_k n_{i,k,m}E_{i,k,m}$.

We perform the above construction until $m = d$ where d is the dimension of V. We claim that:

Lemma 2.6. *Let $C \subset V$ be an irreducible closed subset contained in some $D_{i,j}$. Then for some m, C is not contained in $E_{i,k,m}$ for any i and k.*

Proof. Otherwise, for each m there are $i(m), k(m)$ such that $C \subset E_{i(m),k(m),m}$. Now for each m no component of $D_{i,j} \cap E_{i(1),k(1),1} \cap \ldots \cap E_{i(m-1),k(m-1),m-1}$ is contained in $E_{i(m),k(m),m}$. It follows that for every m every component of $D_{i,j} \cap E_{i(1),k(1),1} \cap \ldots \cap E_{i(m),k(m),m}$ has dimension less than or equal to $d-m-1$. When $m = d$ this is a contradiction. ∎

Now assume $\alpha \in \mathrm{Br}(F(V))$ has exponent q and $\mathcal{D}$ is the set of divisors where α ramifies. Assume we have blown up so that all the elements of $\mathcal{D}$ are non-singular with normal crossings and further that $\mathcal{D}$ is the union of $\mathcal{D}_i$ as in Theorem 2.4. Note that it is possible during the process of blowing up that an exceptional divisor will not be a divisor where α ramifies. If we exclude it from $\mathcal{D}$ and $\mathcal{D}_i$ the conclusions of Theorem 2.4 still stand. Next form the $f_{i,m} \in F(V)$ as in Lemma 2.6 for $1 \leq m \leq d$.

Let F'/F be defined by taking the q roots of all $f_{i,m}$. Note that F'/F has degree less than or equal to q^{d^2}. We see next that F' splits all ramification of α. Note that in [S1, p. 1584] we proved that when q is prime, and S is a non-singular surface, an extension of degree q^2 and not q^4 splits all ramification. It is therefore conceivable that a more careful study of ramification in dimensions greater than 2 would yield a better bound than the one below.

Theorem 2.7 (see [P]). *The restriction $\alpha|_{F'} \in \mathrm{Br}(F')$ is unramified everywhere and the degree of F'/F is bounded by a function of d and q.*

Proof. The second statement is clear. As for the first, suppose S is a discrete valuation domain with $q(S) = F'$. Then S lies over an irreducible closed subset $C \subset V$. Let R be the stalk $\mathcal{O}_{V,C}$ of V at C. If C is not contained in any $D_{i,j}$ then $\alpha \in \mathrm{Br}(R_C)$ and so the restriction of α is unramified at S.

Thus we assume $C \subset D_{i,j}$ for some i, j. Note that by disjointness for each i there is at most one such j. Let I be the set of i where $C \subset D_{i,j}$ for some j. By 2.6 we can take m such that C is not contained in any $E_{i,k,m}$. The $f_{i,m}$, for $i \in I$, must be prime elements of R_C. We are done by Proposition 2.3. ∎

Theorem 2.8. *Suppose $\alpha \in \mathrm{Br}(F_1 \otimes_F F_2)$ has exponent q. Then there are finite field extensions $F'_i \supset F_i$ such that α maps to an everywhere unramified element of $\mathrm{Br}(F'_1 \otimes_F F'_2)$.*

Proof. Since $\alpha \in \mathrm{Br}(F_1 \otimes_F F_2)$, it follows that the ramification locus of α on $V_1 \times_F V_2$ consists of vertical and horizontal irreducible divisors, where "vertical" means the divisor has the form $\pi_1^*(D)$ for $D \subset V_1$ a divisor (and "horizontal" is the V_2 version). We can blow up V_1 so that the vertical irreducible divisors are non-singular and have normal crossings. We do the same thing to V_2. Of course, this implies that their respective pullbacks, taken all together, have non-singular components and normal crossings.

Let $\mathcal{D}_i$ be the irreducible divisors in the ramification locus of α coming from V_i. Let d_i be the dimension of V_i. By further blowing up we may assume that each $\mathcal{D}_i$

is the union of disjoint subsets $\mathcal{D}_{i,j}$ such that the elements of $\mathcal{D}_{i,j}$ are themselves disjoint. Viewing the $\mathcal{D}_i$ as divisors on V_i, form $f_{i,j,m} \in F(V_i)$ as in Lemma 2.6. Let $F'_i \supset F(V_i)$ be the field obtained by adjoining $f_{i,j,m}^{1/q}$ for all j and $1 \leq m \leq d_i$. We can write $F'_i = F(V'_i)$ and $F' = F(V'_1 \times_F V'_2)$.

Let α' be the restriction of α to $\mathrm{Br}(F')$. By 0.3 it suffices to show that α' is unramified with respect to the stalk of any irreducible divisor on $V'_1 \times_F V'_2$. If such a divisor is not horizontal or vertical, then S contains $F'_1 \otimes_F F'_2$ and $\alpha' \in \mathrm{Br}(F'_1 \otimes_F F'_2)$ since $\alpha \in \mathrm{Br}(F_1 \otimes_F F_2)$. Thus we may assume by symmetry that S is the stalk at a vertical divisor $D \times V'_2$. That is, S lies over an irreducible $C \times V_2 \subset V_1 \times V_2$. If R is the stalk $\mathcal{O}_{V_1 \times V_2, C \times V_2}$ then, in the ramification locus of α, only vertical primes appear as primes in R. By 2.2, α is a product of symbols involving vertical primes and an element of $\mathrm{Br}(R_C)$. Thus by the argument of 2.3 and 2.7, if we restrict α to $\mathrm{Br}(L)$ where $L = q(F'_1 \otimes_F F_2)$, then $\alpha|_L$ is unramified at any discrete valuation lying over $C \times V_2$. ■

Section 3. Tensor Products of Fields

In the previous section we saw that the tensor product of fields (over an algebraically closed field) is always a domain. In that sense this is not a case we need to consider. But it will be useful to us, and of considerable interest, to further study tensor products of fields. After all, these rings are the centers of the tensor products of division algebras, and therefore the arithmetic of these rings is important to the study of the more general tensor products.

To begin, in this section F is always an algebraically closed field of characteristic 0. We make this characteristic assumption because we make frequent use of the fact that varieties over F have resolutions of singularities and resolutions of divisors.

Let F_1 and F_2 be fields of finite type over F. That is, the F_i are the fields of fractions of projective F varieties. Because of our assumptions each F_i is, in fact, the function field of a smooth projective variety V_i defined over F. Set $R = F_1 \otimes_F F_2$. Our goal in this section is to study the properties of R and related rings. In particular, we will be interested in the Picard group of R and the Brauer group of R. Let $\bar{F}_i$ denote the algebraic closure of F_i. Frequently we will be extending the scalars of V_i to general $K \supset F$ and more specifically to $\bar{F}_j \supset F_j \supset F$. We write $V_i \times_F K$ as V_i/K and similarly for $\bar{F}_j$. We write $\mathrm{Pic}(V_i/K)$ for the Picard group of V_i/K and $\mathcal{P}ic(V_i/K)$ for the Picard scheme defined over K. As a source for the basic properties of this scheme one can use [BLR, p. 199–235].

By [BLR, p. 232 and p. 210] the connected component $\mathcal{P}ic^0(V_2/K)$ is a projective scheme over K which is of finite type. By [BLR, p. 231] it is smooth and we call it the Picard variety of V_2/K. Being a group scheme, $\mathcal{P}ic^0(V_2/F)$ is an abelian variety. Moreover, $\mathrm{Pic}(V_2/K)$ can be identified with the K points of $\mathcal{P}ic(V_2)$ ([BLR, p. 204]). It therefore makes sense to let $\mathrm{Pic}^0(V_2/K)$ be the K points of $\mathcal{P}ic^0(V_2/F)$. Also, it follows that $\mathrm{Pic}(V_2/F) \to \mathrm{Pic}(V_2/K)$ is injective for any field $K \supset F$. Note that $\mathcal{P}ic(V_2/F) \times_F K = \mathcal{P}ic(V_2/K)$ because of the

functorial definition of $\mathcal{P}ic(V_2/K)$. Because irreducibles over F are absolutely irreducible it follows that $\mathcal{P}ic^0(V_2/F) \times_F K = \mathcal{P}ic^0(V_2/K)$.

Proposition 3.1. *Let K be a field containing F. Then*

$$\operatorname{Pic}(K \otimes_F F_2) = \operatorname{Pic}(V_2/K)/\operatorname{Pic}(V_2/F) = \operatorname{Pic}^0(V_2/K)/\operatorname{Pic}^0(V_2/F).$$

Proof. $\operatorname{Pic}(K \otimes_F F_2)$ is the direct limit of the $\operatorname{Pic}(U/K)$ where $U \subset V_2$ are open subvarieties defined over F. Thus $\operatorname{Pic}(V_2/K) \to \operatorname{Pic}(K \otimes_F F_2)$ is clearly surjective. If some $\alpha \in \operatorname{Pic}(V_2/K)$ maps to 0 in $\operatorname{Pic}(U/K)$, then lifting to divisors we have for some $f \in K(V_2)^*$ that $\alpha = (f) + \sum n_i D_i$ where $V_2 - U$ is the union of irreducibles D_i defined over F. That is, α is in the image of $\operatorname{Pic}(V_2/F)$. This proves the first equality.

Since the irreducible components of $\mathcal{P}ic(V_2/F)$ stay irreducible over K, it follows that $\operatorname{Pic}^0(V_2/K) \to \operatorname{Pic}(V_2/K)/\operatorname{Pic}(V_2/F)$ is surjective and the kernel is $\operatorname{Pic}(V_2/F) \cap \operatorname{Pic}^0(V_2/K) = \operatorname{Pic}^0(V_2/F)$. ■

As the torsion subgroup of $\operatorname{Pic}^0(V_2/K)$ is all defined over F and $\operatorname{Pic}^0(V_2/F)$ is divisible we have:

Corollary 3.2. $\operatorname{Pic}(K \otimes_F F_2)$ *is torsion free.*

Proof. If $\alpha \in \operatorname{Pic}^0(V_2/K)$ satisfies $n\alpha \in \operatorname{Pic}^0(V_2/F)$ then $n\alpha = n\beta$ for some $\beta \in \operatorname{Pic}(V_2/F)$. Thus $n(\alpha - \beta) = 0$ implying $\alpha - \beta \in \operatorname{Pic}(V_2/F)$ and so $\alpha \in \operatorname{Pic}(V_2/F)$. ■

Another immediate corollary of Proposition 3.1. is:

Corollary 3.3. *$\bar{F}_1 \otimes_F F_2$ has divisible Picard group.*

Proof. This follows because $\operatorname{Pic}(\bar{F}_1 \times_F F_2) = \operatorname{Pic}^0(V_2/\bar{F}_1)/\operatorname{Pic}^0(V_2/F)$ and $\operatorname{Pic}^0(V_2/\bar{F}_1)$ is a divisible group. ■

To understand the Brauer group of $F_1 \otimes_F F_2$, we begin by showing that $\bar{F}_1 \otimes_F \bar{F}_2$ has Brauer group 0. The first step is the unramified case.

Lemma 3.4. *Any element in the Brauer group of $V_1 \times V_2$ maps to 0 in $\operatorname{Br}(\bar{F}_1 \otimes \bar{F}_2)$.*

Proof. If $\alpha \in \operatorname{Br}(V_1 \times_F V_2)$, then certainly $\alpha \in \operatorname{Br}(V_2/F_1)$. For any $n > 0$, let $\mu_n \subset \mu$ be the subgroup of roots of 1 of order n. We have (e.g. [M, p. 224]) $H^2(V_2/\bar{F}_1, \mu_n) = H^2(V_2/F, \mu_n)$. The Kummer sequence induces the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Pic}(V_2/F)/n\operatorname{Pic}(V_2/F) & \longrightarrow & H^2(V_2/F, \mu_n) & \longrightarrow & {}_n\operatorname{Br}(V_2/F) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \operatorname{Pic}(V_2/\bar{F}_1)/n\operatorname{Pic}(V_2/\bar{F}_1) & \longrightarrow & H^2(V_2/\bar{F}_1, \mu_n) & \longrightarrow & {}_n\operatorname{Br}(V_2/\bar{F}_1) & \to & 0
\end{array}$$

where ${}_n\operatorname{Br}$ refers to the n torsion. Also we know that $\operatorname{Pic}(V_2/F)/n\operatorname{Pic}(V_2/F) = \operatorname{Pic}(V_2/\bar{F}_1)/n\operatorname{Pic}(V_2/\bar{F}_1)$. Then applying the above diagram for all n we have that $\operatorname{Br}(V_2/F) \to \operatorname{Br}(V_2/\bar{F}_1)$ is an isomorphism. In particular, $\alpha = \alpha_1 + \alpha_2$ where α_1 is in the image of $\operatorname{Br}(V_2/F)$ and α_2 maps to 0 in $\operatorname{Br}(V_2/\bar{F}_1)$. Certainly α_1 maps to 0 in $\operatorname{Br}(\bar{F}_2)$ and so both α_i map to 0 in $\operatorname{Br}(\bar{F}_1 \otimes_F \bar{F}_2)$. ■

By combining Lemma 3.4 with Theorem 2.8 we get:

Theorem 3.5. $\mathrm{Br}(\bar{F}_1 \otimes_F \bar{F}_2) = 0$.

Proof. Any $\bar{\alpha} \in \mathrm{Br}(\bar{F}_1 \otimes_F \bar{F}_2)$ is the image of some $\alpha \in \mathrm{Br}(F(V_1) \otimes_F F(V_2))$. By Theorem 2.8, we may assume that α is in the Brauer group of $V_1 \times_F V_2$, and so we are done by Lemma 3.4. ∎

As a consequence of the above theorem, any element of $\mathrm{Br}(F_1 \otimes_F F_2)$ is split by an extension $F_1' \otimes_F F_2'$ where the F_i'/F_i are Galois with group G_i. That is, $F_1' \otimes_F F_2'$ is Galois over $F_1 \otimes_F F_2$ with group $G_1 \oplus G_2$.

Let $S = F_1' \otimes_F F_2'$ and $R = F_1 \otimes_F F_2$. From [DI, p.116] we know that there is an exact sequence $\mathrm{Pic}(R) \to \mathrm{Pic}(S)^G \to H^2(G, S^*) \to \mathrm{Br}(S/R) \to H^1(G, \mathrm{Pic}(S))$ where $G = G_1 \oplus G_2$.

We next will show that by extending S we may assume that Brauer group elements are crossed products. Let $\bar{G}_i$ be the Galois group of $\bar{F}_i/F_i$ and $\bar{S} = \bar{F}_1 \otimes_F \bar{F}_2$.

Lemma 3.6. *a) Suppose that in the above sequence $\alpha \in \mathrm{Br}(S/R)$. Then there are fields $F_i'' \supset F_i'$ such that F_i''/F_i is Galois with group G_i' mapping to G_i and under restriction α is in the image of $H^2(G_1' \oplus G_2', S'^*)$ where $S' = F_1'' \otimes_F F_2''$.*

b) There is a surjection $H^2(\bar{G}_1 \oplus \bar{G}_2, \bar{S}^*) \to \mathrm{Br}(R)$.

Proof. Part b) is a consequence of a). If $\beta \in H^1(G_1 \oplus G_2, \mathrm{Pic}(S))$ it suffices to show that there are such G_1', G_2', and $S' = F_1'' \otimes_F F_2''$ with β mapping to 0 in $H^1(G_1' \oplus G_2', \mathrm{Pic}(S'))$. Now $H^1(\bar{G}_1 \oplus \bar{G}_2, \mathrm{Pic}(\bar{S}))$ is the direct limit of all such $H^1(G_1' \oplus G_2', \mathrm{Pic}(S'))$ and it suffices to show that $H^1(\bar{G}_1 \oplus \bar{G}_2, \mathrm{Pic}(\bar{S})) = 0$.

By similar reasoning $H^1(\bar{G}_1 \oplus \bar{G}_2, \mathrm{Pic}(\bar{S}))$ is the direct limit of all

$$H^1(\bar{G}_1 \oplus G_2', \mathrm{Pic}(\bar{F}_1 \otimes_F F_2''))$$

taken over all G_2' Galois extensions F_2''/F_2. But by Corollaries 3.2 and 3.3, $\mathrm{Pic}(\bar{F}_1 \otimes_F F_2'')$ is torsion free divisible, so $H^1(\bar{G}_1 \oplus G_2', \mathrm{Pic}(\bar{F}_1 \otimes_F F_2'')) = 0$. ∎

Thus to describe $\mathrm{Br}(F_1 \otimes_F F_2)$, we need to describe $H^2(\bar{G}_1 \oplus \bar{G}_2, \bar{S}^*)$. To this end we next observe:

Lemma 3.7. *a) $S^* \cong (F_1^* \oplus F_2^*)/F^*$ and $\bar{S}^* = (\bar{F}_1^* \oplus \bar{F}_2^*)/F^*$.*

b) $H^2(\bar{G}_1 \oplus \bar{G}_2, \bar{S}^*) \cong H^2(\bar{G}_1 \oplus \bar{G}_2, F^*) \cong H^2(\bar{G}_1 \oplus \bar{G}_2, \mu)$ where $\mu \subset F^*$ is the group of roots of 1, and so $\mu \cong \mathbb{Q}/\mathbb{Z}$ as a $\bar{G}_1 \oplus \bar{G}_2$ module.

Proof. We begin with a). The second statement of a) follows from the first. Suppose $u \in (F_1 \otimes_F F_2)^*$ and let $F_i = F(V_i)$ with V_i projective non-singular. If we consider the principle divisor (u) of u on $V_1 \times V_2$, then all the zeroes and pole components must be horizontal or vertical. Let D be the divisor of vertical zeroes and poles, which we can also view as a divisor of V_1/F. Thus in V_1/F_2, D is a principal divisor, and since $\mathrm{Pic}(V_1/F) \to \mathrm{Pic}(V_1/F_2)$ is injective, we know that D is principal as a divisor over V_1/F. In other words, there is an element $v \in F(V_1)^* = F_1^*$ such that u/v is a unit on V_1/F_2. In other words, $u = vw$ where $v \in F_1^*$ and $w \in F_2^*$. On the other hand, if $vw = 1$, then $v, w \in F_1^* \cap F_2^* = F^*$.

Turning to b), there is an exact sequence

$$0 \to F^* \to (\bar{F}_1^* \oplus \bar{F}_2^*/F^*) \to (\bar{F}_1^*/F^* \oplus \bar{F}_2^*/F^*) \to 0,$$

and each $\bar{F}_i^*/F^*$ is torsion free divisible. 2) follows immediately. ∎

Let $\mathrm{Sym}(\bar{G}_1, \bar{G}_2)$ be defined as the direct limit of $\mathrm{Hom}(\bar{G}_1, \mu_n)\otimes_{\mathbb{Z}}\mathrm{Hom}(\bar{G}_2, \mu_n)$ over all n. Now we can invoke standard group cohomology and observe:

Lemma 3.8. $H^2(\bar{G}_1 \oplus \bar{G}_2, \mu) \cong H^2(\bar{G}_1, \mu) \oplus \mathrm{Sym}(\bar{G}_1, \bar{G}_2) \oplus H^2(\bar{G}_2, \mu)$.

Proof. This follows, for example, from the Hochschild-Serre spectral sequence applied to $\bar{G}_1 \oplus \bar{G}_2 \to \bar{G}_1$. From the product structure, $H^2(\bar{G}_i, \mu) \to H^2(\bar{G}_1 \oplus \bar{G}_2)$ is injective and so $H^2(\bar{G}_2, \mu)^{\bar{G}_1} = H^2(\bar{G}_2, \mu)$ survives unchanged in the limit of the spectral sequence.

It suffices to show that

$$H^1(\bar{G}_1, H^1(\bar{G}_2, \mu)) = \mathrm{Hom}(\bar{G}_1, \mathrm{Hom}(\bar{G}_2, \mu)) = \mathrm{Sym}(\bar{G}_1, \bar{G}_2)),$$

and that all the elements of $\mathrm{Sym}(\bar{G}_1, \bar{G}_2)$ survive in the limit. The first statement follows because $\mathrm{Hom}(\bar{G}_1, \mathrm{Hom}(\bar{G}_2, \mu))$ is the direct limit of the

$$\mathrm{Hom}(\bar{G}_1, \mathrm{Hom}(\bar{G}_2, \mu_n)) = \mathrm{Hom}(\bar{G}_1, \mu_n) \otimes_{\mathbb{Z}} \mathrm{Hom}(\bar{G}_2, \mu_n)$$

for all n. The second statement follows because all the elements of $\mathrm{Sym}(\bar{G}_1, \bar{G}_2)$ are images of cup products of elements of the $H^1(\bar{G}_i, \mu_n)$. ∎

If, as above, the $\bar{G}_i$ are absolute Galois groups of the fields F_i, we write

$$\mathrm{Sym}(\bar{G}_1, \bar{G}_2) = \mathrm{Sym}(F_1, F_2).$$

We can think of this last group as abstract symbols in the cohomology. We are ready for:

Theorem 3.9. $\mathrm{Br}(F_1 \otimes_F F_2) = \mathrm{Br}(F_1) \oplus \mathrm{Br}(F_2) \oplus I$, *where* I *is an image of* $\mathrm{Sym}(F_1, F_2)$.

Proof. By Lemma 3.6 and 3.7 there is a surjection

$$\phi : H^2(\bar{G}_1, \mu) \oplus \mathrm{Sym}(\bar{G}_1, \bar{G}_2) \oplus H^2(\bar{G}_2, \mu) = H^2(\bar{G}_1 \oplus \bar{G}_2, \mu) \to \mathrm{Br}(F_1 \otimes_F F_2).$$

We can identify $H^2(\bar{G}_i, \mu)$ with $\mathrm{Br}(F_i)$. Since $F_i = F(V_i)$ and V_i has an F point, the induced map $\mathrm{Br}(F_i) \to \mathrm{Br}(F_1 \otimes_F F_2)$ is injective. ∎

Note that $\mathrm{Br}(F_1)$ and $\mathrm{Sym}(F_1, F_2)$ map to zero in $\mathrm{Br}(\bar{F}_1 \otimes_F F_2)$ and so taking direct limits we have:

Lemma 3.10. $\mathrm{Br}(\bar{F}_1 \otimes_F F_2) = \mathrm{Br}(F_2)$ *and* $\mathrm{Br}(F_1 \otimes_F \bar{F}_2) = \mathrm{Br}(F_1)$.

Now we can finish 3.9 by noting 3.10 and the restrictions

$$\mathrm{Br}(F_1 \otimes_F F_2) \to \mathrm{Br}(\bar{F}_1 \otimes_F F_2) \qquad \mathrm{Br}(F_1 \otimes_F F_2) \to \mathrm{Br}(F_1 \otimes_F \bar{F}_2)$$

imply that any element in the kernel of ϕ has to be in $\mathrm{Sym}(\bar{G}_1, \bar{G}_2)$. ■

Of course $\mathrm{Sym}(F_1, F_2) \to \mathrm{Br}(F_1 \otimes_F F_2)$ can be interpreted as the union of symbol algebra induced maps $(a_1, a_2) \to (a_1, a_2)_n$ where $a_i \in F_i^*$ but the symbol algebra has center $F_1 \otimes_F F_2$. Also, we know from the example in Section 4 this map is not injective. In fact, noticing the non-injectivity here was the idea that spurred the discovery of the example of section 4. The connection may not be clear, but one way of viewing questions about Schur index over $q(F_1 \otimes_F F_2)$ is that one is considering the "smallest" way of representing a Brauer group element. The non-injectivity above raised the possibility of writing Brauer group elements in terms of fewer symbols by "using" trivial elements in the image of $\mathrm{Sym}(F_1, F_2)$. The connection is perhaps not rigorous, but it was strong enough to suggest the example in the next section.

Section 4. Products of Curves and the Counterexample

In this section we consider Brauer groups over products of curves and use that machinery to provide a counterexample to our main question. Let F be a field of characteristic 0. Although F is not assumed to be algebraically closed, it should be clear that the algebraically closed case provides us important examples.

Suppose C and C' are two curves defined over F with the additional property that all torsion points of the Jacobians $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C')$ are F rational and both curves have F rational points. Let $K = F(C')$ and let $\bar{K}$ be the algebraic closure of K. Let $\bar{C} = C \times_F \bar{K}$, and let G be the Galois group of $\bar{K}/K$. Let $\mathrm{Tor}(\mathrm{Jac}(\bar{C}))$ be the torsion subgroup which by assumption has trivial G action. Since $\mathrm{Jac}(\bar{C})/\mathrm{Tor}(\mathrm{Jac}(\bar{C}))$ is torsion free divisible, every element of $H^1(G, \mathrm{Jac}(\bar{C}))$ is in the image of $H^1(G, \mathrm{Tor}(\mathrm{Jac}(\bar{C}))) = \mathrm{Hom}(G, \mathrm{Tor}(\mathrm{Jac}(\bar{C})))$. We have the following three exact sequences of G modules associated to the curve C.

$$0 \to \bar{K}(C)^*/\bar{K}^* \to \mathrm{Div}(\bar{C}) \to \mathrm{Pic}(\bar{C}) \to 0$$

and

$$0 \to \bar{K}^* \to \bar{K}(C)^* \to \bar{K}(C)^*/\bar{K}^* \to 0$$

and

$$0 \to \mathrm{Jac}(\bar{C}) \to \mathrm{Pic}(\bar{C}) \to \mathbb{Z} \to 0$$

We will frequently apply the long exact cohomology sequence to each of these sequences.

Since C has a K rational point, the last sequence splits. Thus, $\mathrm{Pic}(\bar{C})^G \to \mathbb{Z}$ is surjective, and since $H^1(G, \mathbb{Z}) = 0$ we have that $H^1(G, \mathrm{Jac}(\bar{C})) = H^1(G, \mathrm{Pic}(\bar{C}))$. There is a discrete valuation ring $R \subset K(C)$ with $q(R) = K(C)$ and residue field

$\bar{R} = K$. Thus, $\bar{R} = \bar{K} \otimes_K R$ is a discrete valuation domain with $q(\bar{R}) = \bar{K}(C)$, and $\bar{R} \to \bar{K}$ is a G map. Now $\bar{K}(C)^* = \bar{R}^* \oplus \mathbb{Z}$, so there is a G morphism $\bar{K}(C)^* \to \bar{K}^*$ and the second sequence splits. Thus,

$$H^2(G, \bar{K}(C)^*/\bar{K}^*) = H^2(G, \bar{K}(C)^*)/H^2(G, \bar{K}^*).$$

By Tsen's Theorem (e.g. [Se, p. 162])

$$H^2(G, \bar{K}^*(C)) = \mathrm{Br}(\bar{K}(C)/K(C)) = \mathrm{Br}(K(C))$$

so

$$H^2(G, \bar{K}(C)^*)/\bar{K}^* = \mathrm{Br}(K(C))/\operatorname{Br}(K).$$

For any point P of $\bar{C}$ let G_P be the stabilizer in G of P. Then

$$\mathrm{Div}(\bar{C}) = \oplus_P \ \mathbb{Z}[G/G_P],$$

the direct sum being over all G-orbits of points. Thus

$$H^1(G, \mathrm{Div}(\bar{C})) = \oplus_P \ H^1(G_P, \mathbb{Z}) = 0.$$

Consider the composite $\phi : H^2(G, \bar{K}(C)^*) \to H^2(G, \mathrm{Div}(\bar{C})) = \oplus_P H^1(G_P, \mathbb{Q}/\mathbb{Z})$. Since G_P is the absolute Galois group of the residue field of the C point defined by P, it is easy to see that ϕ is the sum of all the ramification maps at all points of C. It follows that $H^1(G, \mathrm{Jac}(\bar{C})) = H^1(G, \mathrm{Pic}(\bar{C})) \subset \mathrm{Br}(K(C))/\operatorname{Br}(K)$ is the subgroup unramified at all of the points of C, or in different language:

Lemma 4.1. $H^1(G, \mathrm{Jac}(\bar{C})) = \mathrm{Br}(C)/\operatorname{Br}(K)$.

Note that most of the paragraph preceding Lemma 4.1 is essentially due to Roquette ([Ro]).

Recall that we are interested in an element $\alpha \in H^1(G, \mathrm{Jac}(\bar{C}))$ which is the image of an element $\alpha' \in \mathrm{Hom}(G, \mathrm{Tor}(\mathrm{Jac}(\bar{C})))$ with cyclic image of order n. If H is the kernel of α', let $L = \bar{K}^H$. If σH is a generator of G/H, let $P = \alpha'(\sigma)$ be the associated element of order n of $\mathrm{Jac}(\bar{C})$ and let $P' \in \mathrm{Div}(\bar{C})$ be a preimage of P. Let β be the image of α in $H^2(G, \bar{K}(C)^*/\bar{K}^*)$ under the coboundary.

We want to make β more explicit. Since α is the image of some cocycle $\alpha_c \in H^1(G/H, \mathrm{Jac}(\bar{C})^H)$, β is the image of $\beta_c \in H^2(G/H, (\bar{K}(C)^*/\bar{K})^H)$ where β_c is the image of α_c under the G/H coboundary. Since G/H is cyclic, if M is any G/H module, $H^2(G/H, M) \cong M^G/N_{G/H}(M)$ where $N_{G/H} : M \to M$ is the norm map and the isomorphism depends on the choice of generator σH of G/H. In our case $M^G = (\bar{K}(C)^*/\bar{K}^*)^G = K(C)^*/K^*$. Tracing through the G/H coboundary map we see that β_c corresponds to the image of fK^* where $f \in K(C)^*$ is such that the divisor $(f) = nP'$. That is, as a Brauer group element α maps to the cyclic algebra $\Delta(L(C)/K(C), \sigma, f)$ (modulo $\mathrm{Br}(K)$). All together we have:

Lemma 4.2. *The element β, viewed as an element of $\operatorname{Br}(K(C))/\operatorname{Br}(K)$, is represented by the cyclic algebra $\Delta(L(C)/K(C), \sigma, f)$ where L/K, σ and f are as above.*

We are interested in when β is trivial. That is, given $\alpha' : G \to \operatorname{Tor}(\operatorname{Jac}(\bar{C}))$ as above, we are interested when it maps to 0. Let σ be as above.

Lemma 4.3. *The element α' maps to 0 in $H^1(G, \operatorname{Jac}(\bar{C}))$ if and only if there is an L point Q of $\operatorname{Jac}(C)$ with $\sigma(Q) - Q = P$.*

Proof. From the definition of degree 1 cohomology, there is an H fixed Q' in $\operatorname{Jac}(\bar{C})$ such that $\sigma(Q') - Q' = P$. The map $H^2(H, \bar{K}) \to H^2(H, \bar{K}(C))$ has been identified with $\operatorname{Br}(L) \to \operatorname{Br}(L(C))$, which is injective since C has a K rational point. Thus $H^1(H, \bar{K}(C)^*/\bar{K}^*) = 0$ and it follows from the long exact cohomology sequence that Q' is the image of an H fixed element Q'' of the divisor group $\operatorname{Div}(\bar{C})$. That is, as an element of $\operatorname{Pic}(\bar{C})$, Q' can be written as a sum of H orbits of points. In other language, Q' corresponds to an L point, Q, of $\operatorname{Pic}(C)$. After subtracting a suitable multiple of a K rational point, we can assume that this element Q is in the Jacobian and defined over L. ∎

Our goal here is to study elements of the Brauer group of the product of the two curves C and C'. To this end, we now add the assumption that $L = F(C'')$ where $C'' \to C'$ is a cyclic unramfied cover of degree n and C'' has an F rational point. Note that $C'' \to C'$ induces a surjective homomophism $\operatorname{Jac}(\bar{C}'') \to \operatorname{Jac}(\bar{C}')$, and the Galois group of this latter cover is translation by elements in the cyclic kernel of order n. Let $\sigma \in \operatorname{Gal}(L/K) = \operatorname{Gal}(F(C'')/F(C'))$ be a generator. Then $\sigma : C'' \to C''$ induces a covering map $\sigma' : \operatorname{Jac}(C'') \to \operatorname{Jac}(C'')$ and σ' is induced by addition of an order n element $P' \in \operatorname{Jac}(C'')$. Since $P' = \sigma'(0)$, P' is F rational.

The point Q of Lemma 4.3 is $f' : \operatorname{Spec}(F(C'')) \to \operatorname{Spec}(F(C'')) \times_F \operatorname{Jac}(C)$ and $\operatorname{Spec}(F(C'')) \to C''$ induces

$$f' : \operatorname{Spec}(F(C'')) \to C'' \times_F \operatorname{Jac}(C).$$

Let D be the closure of the image of f', and consider the induced map $D \to C''$ which birationally is the identity. Since C'' is the unique non-singular model in $F(C'')$, and there is a desingularization $D' \to D$, it follows that $D' = D$ and $D \to C''$ is the identity. That is, D is the graph of a morphism

$$g : C'' \to \operatorname{Jac}(C).$$

By the universal property of Jacobians, this induces a homomorphism

$$g : \operatorname{Jac}(C'') \to \operatorname{Jac}(C).$$

Since Q is an L point of $\operatorname{Jac}(C)$, it makes sense to form $\sigma(Q)$ which is also a graph of a morphism and in fact $\sigma(Q)$ is the graph of $g' = g \circ \sigma^{-1}$. From

$$\sigma(Q) - Q = P$$

we deduce that as morphisms $g'(x) = g(x) + P$. But σ on $\operatorname{Jac}(C'')$ is translation by P', so as endomorphisms of $\operatorname{Jac}(C'')$ we have $g(x - P') = g(x) + P$, or

$$g(-P') = P.$$

We have shown:

Proposition 4.4. *In the above situation, α is split if and only if there is a homomorphism $g : \mathrm{Jac}(C'') \to \mathrm{Jac}(C)$ such that $g(P') = P$ where $P' \in \mathrm{Jac}(C'')$ generates the kernel of $\mathrm{Jac}(C'') \to \mathrm{Jac}(C')$.*

Let us note a consequence of the above in the case $C = E$ and $C' = E'$ are both elliptic curves.

Lemma 4.5. *If E and E' are not isogenous, then α as above is not split.*

Proof. Of course C'' is also an elliptic curve E'', and $E'' \to E'$ is an isogeny. We can identify E, E' and E'' with their Jacobians and, of course, E' and E'' are isogenous. If α is split, the morphism $g : E'' \to E$ must be an isogeny. ■

Note that Lemma 4.5 is a generalization of [C, p. 138] which states the non-splitting in the case $n = 2$ (but is easily generalized to all n).

Recall that in Lemma 4.2 we wrote α as $\Delta(L/K, \sigma, f)$, where $L = L(C)$ and $K = K(C)$. If F contains a primitive n root of 1, then $L = K(h^{1/n})$. Also, we have assumed L/K is everywhere unramified so the C' divisor of h has the form nE where E is a divisor on C'. Of course E then defines an n torsion point on $\mathrm{Jac}(C')$. Moreover α is the symbol algebra (h, f) of degree n over $F(C' \times C)$.

For our example, we specialize to the case where F is algebraically closed, and C and C' are elliptic curves which we can identify with their Jacobians. Let $n = p$ be a prime. Consider two non-isogenous elliptic curves E and E'. Let P' be an element of order p on E and put $E_2 = E/ < P' >$. Let $Q \in E$ be an independent (of P') element of order p, let $E_3 = E/ < Q >$, and let P be the image of P'. Then $F(E) = F(E_2)(a_2^{1/p})$ and $pP = (a_3)$ so by the above (a_2, a_3) is a split algebra over $F(E_2 \times E_3)$. Similarly we use E' to define E_1 and E_4 and a split algebra (a_1, a_4) over $F(E_1 \times E_4)$. Set $V_1 = E_1 \times E_2$ and $V_2 = E_3 \times E_4$. Now we work over $K = F(V_1 \times V_2) = F(E_1 \times E_2 \times E_3 \times E_4)$ and we view all the a_i as also being elements of K. Set $D_1 = (a_1, a_2)$, a degree p symbol algebra over $F(V_1)$, and $D_2 = (a_3, a_4)$, a degree p symbol algebra over $F(V_2)$.

Theorem 4.6. *Both D_i are division algebras, but $D_1 \otimes_F D_2$ has a zero divisor.*

Proof. Set $D'_i = D_i \otimes_{F(V_i)} K$. It suffices to show that the D_i are division algebras but $D'_1 \otimes_K D'_2$ is not a division algebra. Since E_1 and E_2 are not isogenous, $D_1 = (a_1, a_2)$ is a division algebra over $F(V_1)$ by 4.5. Similarly, D_2 is a division algebra. However, over K, (a_2, a_3) is split so a_2 is a norm from $K(a_3^{1/p})$. Similarly, a_4 is a norm from $K(a_1^{1/p})$. Thus, $K((a_2 a_4)^{1/p})$ is a subfield of both $D'_1 = (a_1, a_2)_K$ and $D'_2 = (a_3, a_4)_K$. Hence $D'_1 \otimes_K D'_2$ is not a division algebra. ■

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